$C^{1,1}$ REGULARITY FOR PRINCIPAL-AGENT PROBLEMS

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ABSTRACT. We prove the interior $C^{1,1}$ regularity of solutions to a subclass of principal-agent problems originally considered by Figalli, Kim, and McCann. Our approach is based on construction of a suitable comparison function which, essentially, allows one to pinch the solution between parabolas. The original ideas for this proof arise from an earlier, unpublished, result of Caffarelli and Lions which we adapt here to general quasilinear benefit functions.

Principal-agent problems are a class of economics models with applications to tax policy, regulation of public utilities, product line design and contract theory; for references see [11, 14, 17]. Mathematically these are optimization problems in the calculus of variations. A subclass of these problems which is economically relevant and mathematically tractable was studied by Figalli, Kim, and McCann [11]. In this article we show solutions to the problems they considered are locally $C^{1,1}$; equivalently, the maps from agent types to products selected will be Lipschitz continuous locally. In this introduction we outline the formulation of, and a bit of history on, the regularity of these problems. Then, in Section 1, we provide precise details and a full formulation of the setting and our results.

We are interested in minimizing a functional

$$L[u] := \int_X F(x, u, Du)\, dx, \tag{1}$$

over the set of b-convex functions. Here b-convexity, is a generalized notion of convexity which arises in the contexts of incentive compatibility [20] and independently, optimal transport [25]; in the bilinear case $b(x, y) = x \cdot y$ of Rochet and Choné [21] it reduces to plain convexity. The quantity of economic interest, the principal's price menu, is a smooth function of u and Du (further details in Section 2). Thus regularity results for u answer questions of economic interest such as "do similar consumers necessarily consume similar products?". (By similar we mean suitably close in the spaces parameterizing consumer and product types respectively.)

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The regularity of minimizers to (1) is complicated by the set of functions over which the applications compel us to minimize. Indeed, even in the simplest case when the set of functions is the cone of convex functions we are prevented from employing the usual techniques in the calculus of variations: Two sided interior perturbations of convex functions are, in general, not convex and, subsequently, it is difficult to derive the Euler–Lagrange equations; however, see Kolesnikov–Sandomirsky–Tsyvinski–Zimin [14], McCann–Zhang [18], Rochet–Choné [21] and their references for recent progress.

The framework in which Figalli, Kim, and McCann worked has links to optimal transport: the mapping between consumers and the product they purchase is the optimal transport map between the density of consumers and the density of products sold. Thus one might hope to bring the regularity theory from optimal transport (namely [16, 12, 24, 10]) to bear on the principal-agent problems. However, apriori we are studying regularity of the optimal transport map in a situation where the target data (density of products sold) is unknown. Thus, once again, the usual approach, via the Monge–Ampère equation, does not apply in our setting. Numerical examples e.g. [19] [2] and theory [1] [21] [18] show the associated data for the Monge–Ampère type equation is singular satisfying neither a lower bound with respect to the Lebesgue measure, nor absolute continuity with respect to the Lebesgue measure.

Because the standard approaches to regularity are unavailable we rely on more bare-handed techniques. We use the only information available — that u is a minimizer to the functional (1). Our proof relies on the construction of a comparison function. This technique, time immemorial in PDEs, allows one to turn the inequality $L[u] \leq L[v]$, which holds for all admissible functions v, into an inequality which directly implies the $C^{1,1}$ regularity.

The literature on these techniques applied to the principal-agent problem is limited. In the special case of a bilinear benefit function the key results are a global C^1 result of Carlier and Lachand–Robert [5], and a interior $C^{1,1}$ result that appears in some unpublished notes by Caffarelli and Lions [3]. Our results and method of proof are based on the work of Caffarelli and Lions. For general quasilinear benefit functions, those related to costs in optimal transport, the C^1 result of Carlier and Lachand–Robert was extended by Chen [8, 7] (who made use of some of Caffarelli and Lions's techniques). In this paper we deal with the same problem that Chen did but prove the analogue of Caffarelli and Lions's $C^{1,1}$ result. We employ, as well as the techniques of the aforementioned works, some results and methods from the optimal transport literature, namely Figalli, Kim, and McCann [11, 12] and also Chen and Wang [9].

Our main result is Theorem 3 stated in Section 1.2 after introducing precisely the setting we're working in. Following this we recall in Section 2 some background material, before proving our key results in Sections 3 and 4.

1. Setting and statement of results

1.1. Economic model

Let us now heuristically explain the economic setting. The mathematical details are given in the next subsection. A monopolist is concerned with selling indivisible products (e.g., cars, houses, smart phones) to consumers. We consider a space of consumers $\mathcal{X} \subset \mathbf{R}^n$ and a space of products $\mathcal{Y} \subset \mathbf{R}^n$. We assume consumers are distributed according to some measure μ on $\mathcal{X}$ and that product y costs the monopolist price $c(y)$. The goal of the monopolist is to assign to each product $y \in \mathcal{Y}$ a price $v(y) \in \mathbf{R}$ in such a way that the monopolist's resulting profit is maximized. The function $v : \mathcal{Y} \to \mathbf{R}$ is called the price menu. Each consumer may purchase at most one product. The benefit a consumer x gains from a product y is given by $b(x, y)$, and thus the utility obtained by consumer x on purchasing product y is $b(x, y) - v(y)$. A rational consumer will maximize their utility. That is, they will purchase the product y which realizes

$$u(x) := \sup_{y\in\mathcal{Y}} b(x, y) - v(y). \tag{2}$$

Employing the notation

$$Yu(x) = \operatorname*{argmax}_{y\in\mathcal{Y}} b(x, y) - v(y) \tag{3}$$

for the product purchased by x, the monopolist's profit is

$$L := \int_{\mathcal{X}} v(Yu(x)) - c(Yu(x))d\mu(x),$$

and their goal is to maximize this quantity over all choices of price menu v. To incentivize the monopolist not to price products too high we assume there is a given "null" product (or "outside option") $y_\emptyset$, which the monopolist is compelled to offer at a fixed price $v(y_\emptyset)$. This results in an additional requirement on the allowable price menus; the resulting utilities must satisfy $u(x) \geq b(x, y_\emptyset) - v(y_\emptyset)$ for all x. We now more precisely define our setting.

1.2. Mathematical formulation

Figalli, Kim, and McCann were able to formulate the above problem using the framework of convexity in optimal transport. This framework strikes a balance between encompassing a broad class of benefit functions $b(x, y)$ whilst allowing the development of a convexity theory.

We take $\mathcal{X}, \mathcal{Y} \subset \mathbf{R}^n$ to be open and bounded. Denote their closures by $\overline{\mathcal{X}}$ and $\overline{\mathcal{Y}}$. We require b is smooth on $\overline{\mathcal{X}} \times \overline{\mathcal{Y}}$. For notation we denote by $b_x(x_0, y)$ the differential of the mapping $x \in \mathcal{X} \mapsto b(x, y)$ evaluated at $x = x_0$, similarly for $b_y(x, y_0)$. We assume the following conditions on b, which we take from [7].

B1. For each $(x_0, y_0) \in \overline{\mathcal{X}} \times \overline{\mathcal{Y}}$ the mappings

$$y \in \overline{\mathcal{Y}} \mapsto b_x(x_0, y)$$
$$\text{and } x \in \overline{\mathcal{X}} \mapsto b_y(x, y_0)$$

are diffeomorphisms onto their ranges.

B2. For each $(x_0, y_0) \in \overline{\mathcal{X}} \times \overline{\mathcal{Y}}$ the sets $b_x(x_0, \mathcal{Y})$ and $b_y(\mathcal{X}, y_0)$ are convex.

For our next condition let

$$Y : \{(x, b_x(x, y)) \ ; \ (x, y) \in \overline{\mathcal{X}} \times \overline{\mathcal{Y}}\} \to \overline{\mathcal{Y}}$$

be defined by

$$b_x(x, Y(x, p)) = p,$$

and note Y is well defined by B1.

B3. For $\xi, \eta \in \mathbf{R}^n$ there holds

$$D_{p_k p_l} b_{x^i x^j}(x, Y(x, p))\xi^i \xi^j \eta_k \eta_l \geq 0.$$

Here we follow the notation of [11], noting that B3 is a variant on the A3 and A3w conditions originally introduced for the regularity theory of optimal maps by Ma, Trudinger, Wang [10] and Trudinger, Wang [15]. The difference is A3 and A3w hold only in the case $\xi \cdot \eta = 0$. The B3 condition is both more common in the economics literature and too more appropriate in that setting: it's a necessary condition for the optimization problem to be convex [11].

For such functions b an associated convexity theory has been developed [11, 16].

Definition 1. *A function $u : \mathcal{X} \to \mathbf{R}$ is called b-convex provided there is $v : \mathcal{Y} \to \mathbf{R}$ such that*

$$u(x) = \sup_{y \in \mathcal{Y}} b(x, y) - v(y). \tag{4}$$

We've already seen that the b-convexity of the consumers utility is a natural consequence of their rationality, as in (2). With this definition (note u, v are finite) a b-convex function has a *support* at each point in its domain. That is, for each $x_0 \in \mathcal{X}$ there is $y_0 \in \overline{\mathcal{Y}}$ such that

$$u(x) \geq u(x_0) + b(x, y_0) - b(x_0, y_0), \qquad \text{for all } x \in \mathcal{X}. \tag{5}$$

By analogy with the bilinear case $b(x, y) = x \cdot y$, we call such lower bounds *b-affine* functions, which in this case forms a *b-support* for u at x_0. For a given x_0 the set of all y_0 such that (5) holds is denoted $Yu(x_0)$. It's a singleton for almost every x. Defined in this way we see that $Yu(x)$ attains the supremum in (6) and therefore agrees with the definition (3). A b-convex function is called uniformly b-convex provided there is $\varepsilon > 0$ such that $D^2 u(x) - b_{xx}(x, Yu(x)) \geq \varepsilon I$. If $b \notin C^2$ we interpret these as directional derivative bounds in the distributional sense. We have a dual notion, b^*-convexity which is obtained by switching the roles of x and y in the above definitions. For example we define b^*-convexity as follows.

Definition 2. *A function* $v : \mathcal{Y} \to \mathbf{R}$ *is called* b^**-convex provided there is* $u : \mathcal{X} \to \mathbf{R}$ *such that*

$$v(y) = \sup_{x \in \mathcal{X}} b(x, y) - u(x). \tag{6}$$

By the equality $u(x) = b(x, Yu(x)) - v(Yu(x))$ we see that the principal-agent problem may be reformulated as follows.

Principal-agent problem Minimize the functional

$$L[u] := \int_{\mathcal{X}} [c(Yu(x)) - b(x, Yu(x)) + u(x)] \, d\mu(x) \tag{7}$$

over the set

$$U_0 := \{u : \mathcal{X} \to \mathbf{R} \, ; \, u \text{ is } b\text{-convex}, u(x) \geq a_\emptyset + b(x, y_\emptyset)\}$$

for a given $a_\emptyset, y_\emptyset$ which arise from the assumption of a "null" product.

If u is the minimizer then the principal's optimal price menu (or its b^*-convex hull) is recovered as $v(y) = \sup_{x \in \mathcal{X}} b(x, y) - u(x)$.

In this setting we can now state our main theorem.

Theorem 3. *Assume b satisfies B1,B2,B3. Assume that c is uniformly* b^**-convex and* $\mu = f \, dx$ *where* $0 < \lambda \leq f(x) \leq \Lambda < \infty$*. Then the solution of the principal-agent problem, u, satisfies* $u \in C^{1,1}_{loc}(\mathcal{X})$ *.*

Remark 4. *Although existence of solutions holds more generally* [6]*, the condition that c is* b^**-convex is standard for the uniqueness theory* [11]*. Moreover a strengthening of convexity to uniform convexity is standard when one moves from uniqueness to regularity (consider for example degenerate ellipticity vs uniform ellipticity arising from convexity properties of a Lagrangian). What's essential is that we obtain the conclusion*

$$c(Y(x, p)) - b(x, Y(x, p)) \tag{8}$$

is uniformly convex in p, independent of x (this is the dual result to Lemma 5 (i)).

2. Background material: convexification and localization

In this section we recall some standard tools for studying b-convexity [9, 11]. Fix a b-convex function $u : \mathcal{X} \to \mathbf{R}$ and $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$. We assume $b_{x^i, y^j}(x_0, y_0) = \delta_{ij}$ which may always be realized after a suitable affine transformation. Define new coordinates

$$\tilde{x}(x) := b_y(x, y_0) - b_y(x_0, y_0), \tag{9}$$

$$\tilde{y}(y) := b_x(x_0, y) - b_x(x_0, y_0). \tag{10}$$

We use the notation $x(\tilde{x})$ to denote the inverse, that is the corresponding x such that for a given $\tilde{x}$ (9) holds. Similarly for $y(\tilde{y})$, for example $x(0) = x_0$ and $y(0) = y_0$. We note all this is well defined by condition B1.

We introduce also the transformed functions

$$\tilde{u}(\tilde{x}) = u(x) - [u(x_0) + b(x, y_0) - b(x_0, y_0)], \tag{11}$$

$$\tilde{b}(\tilde{x}, \tilde{y}) = b(x, y) - [b(x_0, y) + b(x, y_0) - b(x_0, y_0)], \tag{12}$$

for $x = x(\tilde{x})$ and $y = y(\tilde{y})$. The definitions given for b hold for $\tilde{b}$, though $\tilde{b}$ has one less derivative. That is, there is a notion of $\tilde{b}$-convex functions and $\tilde{u}$ defined above is $\tilde{b}$-convex. There is a corresponding $\tilde{Y}\tilde{u}$ mapping and it is straightforward to verify that (with the obvious notation) $\tilde{Y}\tilde{u}(\tilde{x}) = \tilde{y}(Yu(x(\tilde{x})))$.

The following results, using the notation introduced above, provides some tractability to the abstract notions of $\tilde{b}$-convexity theory.

Lemma 5. *[Facts about the $\tilde{b}$-convex geometry]*

(i) The function $\tilde{u} : \tilde{\mathcal{X}} \longrightarrow [0, \infty)$ is convex and $\tilde{u}(0) = 0$. It is uniformly convex if u is uniformly b-convex.

(ii) For all $h \geq 0$ the set

$$S_h := \{\tilde{x} \in \tilde{\mathcal{X}} \; ; \tilde{u}(\tilde{x}) < h\},$$

is convex.

(iii) We have the expansion

$$\tilde{b}(\tilde{x}, \tilde{y}) = \tilde{x} \cdot \tilde{y} + a_{ij,kl}\tilde{x}^i\tilde{x}^j\tilde{y}^k\tilde{y}^l. \tag{13}$$

Here the $a_{ij,kl}$ are smooth functions on $\overline{\mathcal{X}} \times \overline{\mathcal{Y}}$ which arise from taking a Taylor series to fourth order.

These results can be found in, or at least adapted from, [9, 11, 23]. For completeness we include a proof adapted to our setting in Appendix A.

A crucial result in the theory of optimal transport is the following estimate due to Loeper [15], and now commonly called the Loeper maximum principle.

Theorem 6. *(Loeper maximum principle* [15, Theorem 3.2]*) Let $x_0 \in \overline{\mathcal{X}}$ be given. Assume that $(y_t)_{t\in[0,1]}$ is a curve in $\mathcal{Y}$ such that*

$$b_x(x_0, y_t) = (1 - t)b_x(x_0, y_0) + tb_x(x_0, y_1). \tag{14}$$

Then for all $x \in \overline{\mathcal{X}}$ and $t \in [0, 1]$ there holds

$$b(x, y_t) - b(x_0, y_t) \leq \max\{b(x, y_0) - b(x_0, y_0), b(x, y_1) - b(x_0, y_1)\}.$$

Loeper's result was proved under A3w, however under the current hypotheses (B3 as opposed to A3w), the function $t \in [0, 1] \mapsto b(x, y_t) - b(x_0, y_t)$ is in fact convex [13]. We remark that when (14) holds $(y_t)_{t\in[0,1]}$ is called the b^*-segment joining y_0 to y_1 with respect to x_0.

3. Proof of Theorem 3 assuming a key lemma

In this section we state Lemma 7 which, whilst not of independent interest, simplifies the proof of Theorem 3. Indeed, following the statement of Lemma 7 we immediately prove Theorem 3 and later, in Section 4, prove Lemma 7.

Lemma 7. *Let u be a b-convex function, $x_0 \in \mathcal{X}$ and $y_0 \in Yu(x_0)$. Assume r is less than a given r_0 depending only on b and*

$$h := \sup_{B_r(x_0)} [u(x) - (b(x, y_0) - b(x_0, y_0) + u(x_0))],$$

is positive. Then, there is, a b-affine function $p_y(\cdot) := b(\cdot, y) + a$, and corresponding section $S := \{x \in \mathcal{X} \ ; \ u(x) < p_y(x)\}$ which has positive measure and is contained in a slab of width $C_1 r$, such that

$$\sup_{x \in S}[p_y(x) - u(x)] \leq h, \tag{15}$$

$$\int_S \big(c(y) - b(x, y)\big) - \big(c(Yu(x)) - b(x, Yu(x))\big) \, f(x) \, dx \tag{16}$$

$$\leq C_2 h|S| - C_3 \frac{h^2}{r^2}|S|,$$

for $C_1, C_2, C_3 > 0$ depending on b, f, $dist(x, \partial\mathcal{X})$.

Using Lemma 7, Theorem 3 has the following short proof.

Proof. (Theorem 3). Fix arbitrary $x_0 \in \mathcal{X}$ and $y_0 \in Yu(x_0)$. Let

$$p_0(\cdot) := b(\cdot, Yu(x_0)) - b(x_0, Yu(x_0)) + u(x_0)$$

be a b-support at x_0. Since the cost function is C^4, it's standard [4, Proposition 1.2] that to prove u is $C^{1,1}$ it suffices to prove that for all $r \leq r_0$ there holds

$$\sup_{B_r(x_0)} |u - p_0| \leq Cr^2.$$

Of course $u - p_0 \geq 0$; we must prove $u - p_0 \leq Cr^2$. Here's where the lemma enters. Put

$$h = \sup_{B_r(x_0)} u - p_0,$$

and note if $h = 0$, we're done. Thus, we assume $h > 0$ and take the b-affine function p_y and associated section S given by Lemma 7. We set

$$u_h = \max\{u, p_y\}.$$

Note u_h is admissible for the Monopolist's problem: it is b- convex and not less than u. Thus, since u is a minimizer for the principal-agent problem

$$0 \leq L[u_h] - L[u].$$

Because u_h differs from u only on S we compute $L[u_h] - L[u]$ and obtain

$$0 \leq \int_S \Big[\big(c(y) - b(x, y) + p_y\big) - \big(c(Yu(x)) - b(x, Yu(x)) + u\big)\Big] \, f(x) \, dx.$$

From (15) and (16) it's immediate that this inequality becomes

$$0 \leq Ch|S| - C_2|S|\frac{h^2}{r^2}.$$

This gives $h \leq Cr^2$ to establish the theorem. ∎

4. Proof of Lemma 7: Section containment and energy estimate

In the remainder of this paper we prove Lemma 7.

Proof. (Lemma 7). Step 1. (Setup) Chen showed $u \in C^1$ [8, 7]. Alternately, to make our proof logically independent of his, one can use an approximation procedure, standard in the regularity theory for optimal transport, to assume u is C^1. Indeed for possibly non-differentiable u, [15, Theorem 3.1] yields a sequence of C^1 b-convex functions u_k which converge to u in the topology of local uniform convergence. Proving the Lemma for the u_k, then taking a limit yields the result for u. We note the limiting procedure uses that C_1, C_2 may be taken independent of k and the dominated convergence theorem, along with assumed estimates for $|c|, |b|, |f|$.

Working under the assumptions of Lemma 7 we fix x_0 where without loss of generality $x_0 = 0$ and set $y_0 = Yu(x_0)$. Thus we can perform the change of variables and transformations (9)–(12), afterwhich the support at 0 is 0. By direct calculations we see if Lemma 7 holds for transformed quantities it holds for the original functions and coordinates, though with different constants C_1, C_2. Thus it suffices to prove the lemma after applying the transformations (9)–(12). For ease of notation we retain x, y, u, b, that is, we don't switch to the notation $\tilde{x}, \tilde{y}, \tilde{u}, \tilde{b}$ — though keep in mind now b satisfies (13).

Now, without loss of generality $h := \sup_{B_r} u$ occurs at re_1. Then because u attains a maximum over ∂B_r at re_1 it has 0 tangential derivative, that is $Du(re_1) = \kappa e_1$ for some κ. The convexity of (the transformed function) u implies $\kappa \geq h/r$. Since the gradient of the b-support at re_1 agrees with the gradient of u, $y_1 := Yu(re_1)$ satisfies $b_x(re_1, y_1) = \kappa e_1$. Moreover the transformed b satisfies $b_x(re_1, 0) = 0$. Therefore, by considering the b^*-segment joining y_1 to 0 with respect to re_1, we obtain $y_{1/2}$ satisfying

$$b_x(re_1, y_{1/2}) = \frac{h}{2r} e_1. \tag{17}$$

The desired b-affine function will be $p_{y_{1/2}}(x) := b(x, y_{1/2}) - b(re_1, y_{1/2}) + u(re_1)$. Because $p_{y_{1/2}}(0) - u(0) > 0$ (see (24)) we obtain that $S := \{x ; u(x) < p_{y_{1/2}}(x)\}$ has positive measure.

Step 2. (Proof of (15)*)* Since $p_{y_1}(x) := b(\cdot, y_1) - b(re_1, y_1) + u(re_1)$ is a support at re_1 there holds $b(\cdot, y_1) - b(re_1, y_1) = p_{y_1} - h \leq u$. The Loeper maximum principle (Theorem 6) implies for all $x \in \mathcal{X}$

$$b(x, y_{1/2}) - b(re_1, y_{1/2}) \leq \max\{0, b(x, y_1) - b(re_1, y_1)\} \leq u(x). \tag{18}$$

Here we've applied the Loeper maximum principle noting $0 = b(\cdot, 0) - b(re_1, 0)$ is also a support after the transformation (12). Estimate (15) is obtained from (18) by adding $h = u(re_1)$ and subtracting $u(x)$.

Step 3. (Basic estimates on $y_{1/2}$) The rest of the proof, in which we obtain (16), is based primarily on estimates employing (13). Key to this result will be establishing the slab-containment condition

$$S = \{x \,;\, u(x) < p_{y_{1/2}}(x)\} \subset \{x \,;\, |x^1| \leq Cr\} \tag{19}$$

for C depending only on b and diam$\mathcal{X}$ and sup $|u|$.

We begin with basic estimates showing $y_{1/2}$ is a small perturbation of $\frac{h}{2r}e_1$. By (17) and the expansion (13) we have

$$\frac{h}{2r}e_1 = b_x(re_1, y_{1/2}) = y_{1/2} + f_{ijk}(re_1)^i(y_{1/2})^j(y_{1/2})^k, \tag{20}$$

for f_{ijk} a suitable smooth function arising from taking derivatives of (13). Choosing r sufficiently small (depending only on b and its domain) and setting $\varepsilon = rC(n)\sup|f_{ijk}|$ for $C(n)$ a suitable constant depending only on n we have

$$\left|\frac{h}{2r}e_1 - y_{1/2}\right| \leq \varepsilon|y_{1/2}|^2. \tag{21}$$

An inequality of this form implies[1] if θ is the angle $y_{1/2}$ makes with the e_1 axis, then $\sin\theta \leq \varepsilon|y_{1/2}|$ (see Figure 1(a)).

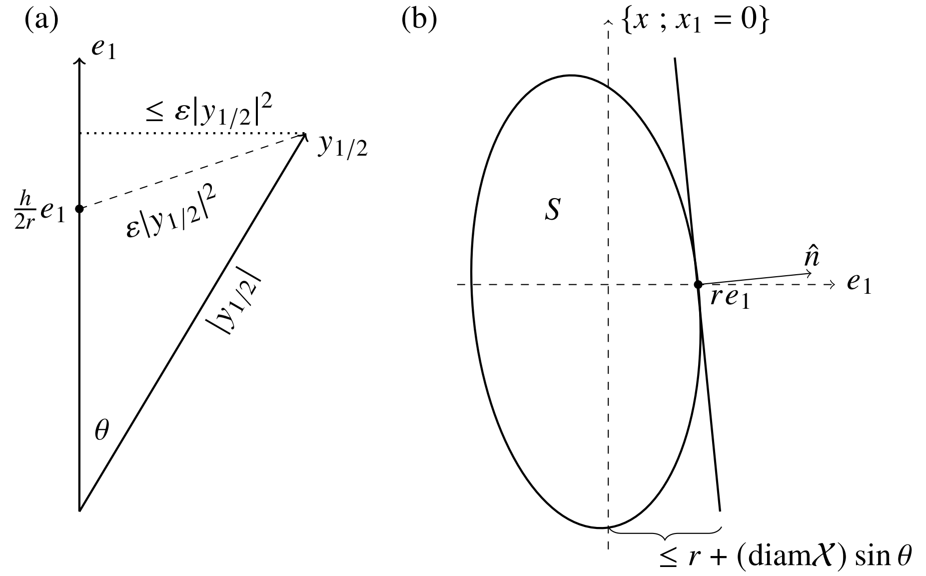

FIGURE 1. Two geometric arguments. In (a) we illustrate how the estimate $\left|\frac{h}{2r}e_1 - y_{1/2}\right| \leq \varepsilon|y_{1/2}|^2$ yields the esimate $\sin\theta \leq \varepsilon|y_{1/2}|$ (recall trigonometry). In (b) we illustrate that if the normal, $\hat{n}$, to a support plane to the convex body S makes angle $\sin\theta \leq Cr$ with the e_1 axis $S \subset \{x \,;\, x^1 \leq r + (\sin\theta)\mathrm{diam}(\mathcal{X}) \leq Kr\}$.

We perform a further change of coordinates to the coordinates in which sections with respect to $p_{y_{1/2}}$ are convex. That is, set

$$\tilde{x}(x) := b_y(x, y_{1/2}), \tag{22}$$

[1]We take these types of arguments from [9].

and consider $\overline{u}, \overline{p}$ defined by $\overline{u}(\tilde{x}) := u(x)$ and $\overline{p}_{y_{1/2}}(\tilde{x}) := p_{y_{1/2}}(x)$ for the unique x such that $\tilde{x}(x) = \tilde{x}$. We do not transform the y-coordinates. We note that in these coordinates the sets

$$\tilde{S} := \{\tilde{x} \,;\, \overline{u}(\tilde{x}) \leq \overline{p}_{y_{1/2}}(\tilde{x})\}$$
$$\text{and} \quad \tilde{S_0} := \{\tilde{x} \,;\, 0 \leq \overline{p}_{y_{1/2}}(\tilde{x})\},$$

are convex by Lemma 5 (ii).

Step 4. (The containment condition (19)*)* Our goal now is to establish (19). In fact, we only need this estimate in the $\tilde{x}$ coordinates, so it is in that setting we will prove it. We begin with the observation that $\tilde{S} \subset \tilde{S}_0$ hence $\tilde{S}_0^c \subset \tilde{S}^c$, where ^{c} denotes complementation. This follows from $\overline{u} \geq 0$. Thus to prove (19) it suffices to prove $\tilde{S} \subset \{\tilde{x} \,;\, \tilde{x}^1 \leq Cr\}$ and $\tilde{S}_0 \subset \{\tilde{x} \,;\, \tilde{x}^1 \geq -Cr\}$. Let's start with $\tilde{S} \subset \{\tilde{x} \,;\, \tilde{x}^1 \leq Cr\}$. Note that $\tilde{x}(re_1)$ is in ∂S since $\overline{u}(\tilde{x}(re_1)) = \overline{p}_{y_{1/2}}(\tilde{x}(re_1))$. Moreover the gradient to the function defining the section, that is

$$D_{\tilde{x}}(\overline{u} - \overline{p}_{y_{1/2}})(\tilde{x}(re_1)),$$

will be a normal to a supporting hyperplane of $\tilde{S}$. By the chain rule

$$D_{\tilde{x}}(\overline{u} - \overline{p}_{y_{1/2}})(\tilde{x}(re_1)) = [b_{xy}(re_1, y_{1/2})]^{-1} D_x(u - p_{y_{1/2}})(re_1). \tag{23}$$

By the expansion (13) we have that, for suitable continuous g_i,

$$b_{xy}(re_1, y_{1/2}) = I + r g_i(re_1, y_{1/2})(y_{1/2})^i = I + O(r)$$

where $O(r)$ here denotes a matrix for which each component is $O(r)$ as $r \to 0$. The same is true for the inverse; $[b_{xy}(re_1, y_{1/2})]^{-1} = I + O(r)$. Thus by employing also (17), we see (23) becomes

$$D_{\tilde{x}}(\overline{u} - \overline{p})(\tilde{x}(re_1)) = \left(\kappa - \frac{h}{2r}\right) e_1 + O(r) \left(\kappa - \frac{h}{2r}\right) e_1.$$

The same argument used after (21) implies that $D_{\tilde{x}}(\overline{u} - \overline{p}_{y_{1/2}})(\tilde{x}(re_1))$, which is the normal to the supporting hyperplane of $\tilde{S}$ at $\tilde{x}(re_1)$, makes angle $\overline{\theta}$ with the e_1 axis for $\sin\overline{\theta} \leq Cr$. Now because the supporting hyperplane passes through $\tilde{x}(re_1) = re_1 + O(r^2)$ and makes angle $\sin\overline{\theta}$ with the plane $\tilde{x}^1 = 0$ we have

$$\tilde{S} \subset \{\tilde{x} \,;\, \tilde{x}^1 \leq r + O(r^2) + C(\sin\overline{\theta})\mathrm{diam}(\tilde{S}) \leq Cr\}.$$

A similar argument will be used for the estimate

$$\tilde{S_0} \subset \{\tilde{x} \,;\, \tilde{x}^1 \geq -Cr\}.$$

First note, using (13) and (20), we obtain

$$\begin{aligned}
p_{y_{1/2}}(0) &\geq -y_{1/2} \cdot (re_1) - Cr^2|y_{1/2}|^2 + h \\
&\geq \left(\frac{-h}{2r}e_1\right) \cdot (re_1) - C|r|^2|y_{1/2}|^2 + h \\
&\geq \frac{-h}{2} - Ch^2 + h > \frac{h}{4}. \qquad (24)
\end{aligned}$$

In the final line we've used $|y_{1/2}| \leq C\frac{h}{r}$ and that $Ch^2 \leq h/4$ provided h is sufficiently small (which is assured by r sufficiently small and the locally-Lipschitz property of u). On the other hand, similar reasoning yields

$$\begin{aligned}
p_{y_{1/2}}(-2re_1) &= b(-2re_1, y_{1/2}) - b(re_1, y_{1/2}) + h \\
&\leq (-2re_1) \cdot y_{1/2} - (re_1) \cdot y_{1/2} + h + C|r|^2|y_{1/2}|^2 \\
&\leq (-2r)\left(\frac{h}{2r}\right) - r\left(\frac{h}{2r}\right) + h + C|r|^2|y_{1/2}|^2 < \frac{h}{4}
\end{aligned}$$

Thus in the x coordinates S_0 has a boundary point $-te_1$ for some $t \in (0, -2r)$ and subsequently in the $\tilde{x}$ coordinates $\tilde{S_0}$ has boundary point $\tilde{x}(-te_1) = -te_1 + O(t^2)$. Similar techniques to above yield that at this boundary point, $\tilde{x}(-te_1)$, the outer normal $D_{\tilde{x}}\overline{p}$ makes angle $\sin\underline{\theta} \leq Cr$ with the negative e_1 axis. The same tangent hyperplane argument as above, this time applied to the convex set S_0, implies $\tilde{S_0} \subset \{\tilde{x}\ ;\ \tilde{x}^1 \geq -Cr\}$. This completes the proof of (19), the containment condition.

Step 5. (Proof of (16)*)* Now we start the proof of (16). Our argument relies on (19). Thus we prove (16) after the further change of variables (22). For brevity we retain the $x, b, p_{y_{1/2}}$ notation (no tilde). The estimate (16) will then hold for the original coordinates and functions, with possibly different constants C_1, C_2.

Now, the uniform convexity condition on c noted in Remark 4 implies

$$\begin{aligned}
\int_S \big[c(Yu(x)) &- b(x, Yu(x)) - c(y_{1/2}) + b(x, y_{1/2})\big]\, f(x)\, dx \\
&= \int_S \big[c\big(Y(x, Du(x))\big) - b\big(x, Y(x, Du(x))\big) \\
&\qquad - c\big(Y(x, Dp_{y_{1/2}}(x))\big) + b\big(x, Y(x, Dp_{y_{1/2}}(x))\big)\big] f(x)\, dx \\
&\geq \int_S \big[\varepsilon|Dp_{y_{1/2}}(x) - Du(x)|^2 \qquad (25) \\
&\qquad + D_p[c(y_{1/2}) - b(x, y_{1/2})] \cdot \big(Du(x) - Dp_{y_{1/2}}(x)\big)\big] f(x)\, dx,
\end{aligned}$$

where ε derives from the uniform convexity and derivatives with respect to p are meant in the sense

$$D_p[c(y_{1/2}) - b(x, y_{1/2})] = D_p[c(Y(x,p)) - b(x, Y(x,p))]|_{p=b_x(x, y_{1/2})}.$$

We compute estimates for both terms in (25). Using the divergence theorem

$$
\begin{aligned}
&\int_S D_p[c(y_{1/2}) - b(x, y_{1/2})] \cdot \big(Du(x) - Dp_{y_{1/2}}(x)\big) f(x)\, dx \\
&\quad = \int_S \nabla_x \cdot \Big((u - p_{y_{1/2}}) f(x) D_p[c(y_{1/2}) - b(x, y_{1/2})]\Big) \\
&\qquad\quad - (u - p_{y_{1/2}}) \nabla_x \cdot \big[f(x) D_p\big(c(y_{1/2}) - b(x, y_{1/2})\big)\big]\, dx \\
&\quad \geq \int_{\partial S} f(x)(u - p_{y_{1/2}}) D_p[c(y_{1/2}) - b(x, y_{1/2})] \cdot \hat{n}\, ds - C|h||S|. \qquad (26)
\end{aligned}
$$

The final inequality uses (15) to control $p_{y_{1/2}} - u$. To compute the boundary integral decompose ∂S as $\partial S = (\partial S \cap \mathcal{X}) \cup (\partial S \cap \partial \mathcal{X})$. Then $p_{y_{1/2}} - u$ is 0 on $\partial S \cap \mathcal{X}$ and bounded by h on $(\partial S \cap \partial \mathcal{X})$ so (26) becomes

$$
\begin{aligned}
&\int_S D_p[c(y_{1/2}) - b(x, y_{1/2})] \cdot \big(Du(x) - p_{y_{1/2}}(x)\big)(x)\, dx \\
&\qquad \geq -Ch|\partial S \cap \partial \mathcal{X}| - Ch|S| \geq -Ch|S|. \qquad (27)
\end{aligned}
$$

Here the final inequality is by the estimate $|\partial S \cap \partial \mathcal{X}| \leq C|S|$ relating the area of $\partial S \cap \partial \mathcal{X}$ to the volume of S, proved by Carlier and Lachand-Robert [5] and Chen [7, pg. 47].

To estimate the remaining term in (25), that is the term

$$\int_S \varepsilon |Dp_{y_{1/2}}(x) - Du(x)|^2 f(x)\, dx,$$

we follow the argument of Caffarelli and Lions. We let $P(x) = (0, x')$ for $x = (x^1, x')$ be the projection onto $\{x \,;\, x^1 = 0\}$. We also denote by $S/2$ the dilation of S by a factor $1/2$ with respect to the origin. Now for each $(0, x') \in P(S/2)$ we denote by $l_{x'}$ the line segment

$$l_{x'} := (P^{-1}(x') \cap S \cap \{x \,;\, x^1 \geq 0\}) \setminus (S/2),$$

and write $l_{x'} = [a_{x'}, b_{x'}] \times \{x'\}$ where $b_{x'} > a_{x'}$. We claim the following

$$u((a_{x'}, x')) - p_{y_{1/2}}((a_{x'}, x)) \leq -h/8, \qquad (28)$$

$$u((b_{x'}, x')) - p_{y_{1/2}}((b_{x'}, x)) = 0, \qquad (29)$$

$$d_{x'} := b_{x'} - a_{x'} \leq Cr. \qquad (30)$$

Here (28) is by the convexity of $u - p_{y_{1/2}}$ along a line segment joining the origin, where $u - p_{y_{1/2}} \leq -h/4$, to $(2a_{x'}, 2x') \in \partial S$, where $u - p_{y_{1/2}} = 0$. Then (29) is by $(b_{x'}, x') \in \partial S$ and (30) is by the containment condition (19).

Thus, by an application of Jensen's inequality we have

$$\begin{aligned}
&\int_{a_{x'}}^{b_{x'}} [D_{x^1} p_{y_{1/2}}((t,x')) - D_{x^1} u((t,x'))]^2 dt \\
&\quad \geq \frac{1}{d_{x'}} \left(\int_{a_{x'}}^{b_{x'}} D_{x^1} p_{y_{1/2}}((t,x')) - D_{x^1} u((t,x')) dt \right)^2 \\
&\quad \geq \frac{1}{d_{x'}} \left(\frac{h}{8} \right)^2 \geq Ch^2/r. \qquad (31)
\end{aligned}$$

To conclude we integrate along all lines $l_{x'}$ for $x' \in P(S/2)$. Indeed

$$\begin{aligned}
\int_S |Dp_{y_{1/2}} - Du|^2 \, dx &\geq \int_{P(S/2)} \int_{a_{x'}}^{b_{x'}} |D_{x^1} p_{y_{1/2}} - D_{x^1} u|^2 \, dt \, dx' \\
&\geq \int_{P(S/2)} Ch^2/r \, dx' \\
&= C \frac{h^2}{r^2} (r|P(S/2)|) \\
&\geq C \frac{h^2}{r^2} |S|. \qquad (32)
\end{aligned}$$

Here the final inequality uses the convexity of S and the containment condition (19). Substituting (27) and (32) into (25) (recall $f \geq \lambda$) yields the energy estimate (16), thereby completing the proof of Lemma 7. ■

Appendix A. Proof of Lemma 5

Here we prove Lemma 5. Whilst results like these appear in a number of works, the techniques we employ in the proof are taken from [13, 12, 23, 9, 22].

For parts (i) and (ii) we follow [22, 23]. Set

$$h(\tilde{x}) := u(x) - b(x, y_0),$$

for x such that $\tilde{x}(x) = x$. We compute an expression for the Hessian of h. By direct calculation

$$D_{\tilde{x}^i \tilde{x}^j} h = [u_{x^\alpha x^\beta}(x) - b_{x^\alpha x^\beta}(x, y_0)] \frac{\partial x^\alpha}{\partial \tilde{x}^i} \frac{\partial x^\beta}{\partial \tilde{x}^j} + [u_\alpha(x) - b_\alpha(x, y_0)] \frac{\partial^2 x^\alpha}{\partial \tilde{x}^i \partial \tilde{x}^j}.$$

The b-convexity of u implies

$$u_{x^\alpha x^\beta}(x) \geq b_{x^\alpha x^\beta}(x, Y(x, Du)) + \varepsilon \delta_{ab}, \qquad (33)$$

where $\varepsilon > 0$ if b is uniformly b-convex and $\varepsilon = 0$ if u is merely b-convex. Thus

$$\begin{aligned}
D_{\tilde{x}^i \tilde{x}^j} h = {} & [b_{x^\alpha x^\beta}(x, Y(x, p_1)) - b_{x^\alpha x^\beta}(x, Y(x, p_0))] \frac{\partial x^\alpha}{\partial \tilde{x}^i} \frac{\partial x^\beta}{\partial \tilde{x}^j} \\
& + [p_1 - p_0]_k \frac{\partial^2 x^k}{\partial \tilde{x}^i \partial \tilde{x}^j} + \varepsilon \sum_\alpha \frac{\partial x^\alpha}{\partial \tilde{x}^i} \frac{\partial x^\alpha}{\partial \tilde{x}^j}, \qquad (34)
\end{aligned}$$

where we've set $p_1 = Du(x), p_0 = b_x(x, y_0)$. A direct calculation implies

$$\frac{\partial^2 x^k}{\partial\tilde{x}^i\partial\tilde{x}^j} = -D_{p_k} b_{x^\alpha x^\beta}(x, Y(x, p_0))\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\beta}{\partial\tilde{x}^j}.$$

Thus (34) may be rewritten as

$$\begin{aligned} D_{\tilde{x}^i\tilde{x}^j} h \geq\ &[b_{x^\alpha x^\beta}(x, Y(x, p_1)) - b_{x^\alpha x^\beta}(x, Y(x, p_0)) \\ &- (p_1 - p_0)_k D_{p_k} b_{x^\alpha x^\beta}(x, Y(x, p_0))]\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\beta}{\partial\tilde{x}^j} + \varepsilon\sum_\alpha\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\alpha}{\partial\tilde{x}^j}. \end{aligned}$$

Using a Taylor series for the function $p \mapsto b_{x^\alpha x^\beta}(x, Y(x, p))$ we obtain

$$D_{\tilde{x}^i\tilde{x}^j} h \geq D_{p_k p_l} b_{x^\alpha x^\beta}(x, Y(x, p_\tau))\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\beta}{\partial\tilde{x}^j}(p_1 - p_0)_k(p_1 - p_0)_l + \varepsilon\sum_\alpha\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\alpha}{\partial\tilde{x}^j}.$$

Now we test the convexity in an arbitrary direction. Fix $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ and set $\tilde{h}(t) = h(t\tilde{x}_1 + (1 - t)\tilde{x}_0)$. The above expression and B3 implies

$$\tilde{h}''(t) \geq \varepsilon\sum_\alpha\frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial x^\alpha}{\partial\tilde{x}^j}(\tilde{x}_1 - \tilde{x}_0)_i(\tilde{x}_1 - \tilde{x}_0)_j. \tag{35}$$

To estimate this from below we compute $\frac{\partial x}{\partial\tilde{x}} = [b_{x,y}(x, y_0)]^{-1}$ and by condition B1, $|\det[b_{x,y}(x, y_0)]^{-1}| > c_0$ for some positive c_0. Thus there is c_1 such that for all unit vectors ξ we have $|[b_{x,y}(x, y_0)]^{-1}\xi| \geq c_1$ (indeed, $[b_{x,y}(x, y_0)]^{-1}\xi \neq 0$ and compactness implies the lower bound). Note $\frac{\partial x^\alpha}{\partial\tilde{x}^i}(\tilde{x}_1 - \tilde{x}_0)_i$ is the α^{th} component of the vector $[b_{x,y}(x, y_0)]^{-1}(\tilde{x}_1 - \tilde{x}_0)$. Subsequently (35) becomes

$$\tilde{h}''(t) \geq \varepsilon|\tilde{x}_1 - \tilde{x}_0|^2.$$

This implies the corresponding uniform convexity of $\tilde{u}$ when $\varepsilon > 0$ and convexity when $\varepsilon = 0$.

Note this inequality also implies point (ii) in Lemma 5. Indeed we assume in addition $\tilde{x}_0, \tilde{x}_1 \in S_h$. By our previous computations $\tilde{h}$ is convex in t. Moreover $\tilde{h}(0), \tilde{h}(1) \leq h$ yielding the same for $t \in (0, 1)$: $\tilde{h}(t) \leq h$, which is convexity of S_h.

For result (iii) in Lemma 5 we assume $b_{ij}(x_0, y_0) = \delta_{ij}$ which can always be satisfied after an initial affine transformation and $(x_0, y_0) = (0, 0)$. Then by a second-order Taylor series for $\tilde{b}$ we have

$$\tilde{b}(\tilde{x}, \tilde{y}) = \tilde{b}(0, \tilde{y}) + \tilde{b}_{\tilde{x}^i}(0, \tilde{y})\tilde{x}^i + \frac{1}{2}\tilde{b}_{\tilde{x}^i\tilde{x}^j}(\tilde{x}^t, \tilde{y})\tilde{x}^i\tilde{x}^j. \tag{36}$$

From (12), $\tilde{b}(0, y) = 0$ is clear. In addition

$$\tilde{b}_{\tilde{x}^i}(0, \tilde{y}) = [b_{x^k}(x_0, y) - b_{x^k}(x_0, y_0)]\left(\frac{\partial x^k}{\partial\tilde{x}^i}\Big|_{\tilde{x}=0}\right) = \tilde{y}^i. \tag{37}$$

Another Taylor series, implies

$$\begin{aligned}\tilde{b}_{\tilde{x}^i\tilde{x}^j}(\tilde{x}_t,\tilde{y}) &= \tilde{b}_{\tilde{x}^i\tilde{x}^j}(\tilde{x}_t,0)+\tilde{b}_{\tilde{x}^i\tilde{x}^j\tilde{y}^k}(\tilde{x}_t,0)\tilde{y}_k+\frac{1}{2}\tilde{b}_{\tilde{x}^i\tilde{x}^j\tilde{y}^k\tilde{y}^l}\tilde{y}^k\tilde{y}^l\\ &= \frac{1}{2}\tilde{b}_{\tilde{x}^i\tilde{x}^j\tilde{y}^k\tilde{y}^l}\tilde{y}^k\tilde{y}^l. \qquad (38)\end{aligned}$$

Here we've used that $\tilde{b}(\tilde{x},0)=0$ and, similarly to (37), $\tilde{b}_{\tilde{y}^k}(\tilde{x},0)=\tilde{x}^k$, whereby the third derivative term vanishes. Equations (37) and (38) into (36) yields Lemma 5 (iii).

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