Monge-Kantorovich Duality in Optimal Transport with Nonadditive Measures on Finite Spaces

Kelvin Shuangjian Zhang

University of Waterloo

http://shuangjian.info/

joint work with Mario Ghossoub and David Saunders

Optimization under Ambiguity and Applications to Finance and Insurance
INFORMS Annual Meeting 2022

October 18, 2022

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces
- Optimal Transport with Nonadditive Measure marginals

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces
- Optimal Transport with Nonadditive Measure marginals
- Explicit solutions

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces
- Optimal Transport with Nonadditive Measure marginals
- Explicit solutions
- Cores

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces
- Optimal Transport with Nonadditive Measure marginals
- Explicit solutions
- Cores
- Duality results

Outline

- Transferable Utility Matching and Optimal Transport on finite spaces
- Optimal Transport with Nonadditive Measure marginals
- Explicit solutions
- Cores
- Duality results
- Conclusion

Transferable Utility Matching and Optimal Transport

[1] Monge (1781)

Transferable Utility Matching and Optimal Transport

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty finite sets, u and v be probability measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi_a(u, v)$ the set of measures on $\mathcal{X} \times \mathcal{Y}$ that has the marginals u on $\mathcal{X}$ and v on $\mathcal{Y}$. That is,

[1]Monge (1781)

Transferable Utility Matching and Optimal Transport

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty finite sets, u and v be probability measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi_a(u, v)$ the set of measures on $\mathcal{X} \times \mathcal{Y}$ that has the marginals u on $\mathcal{X}$ and v on $\mathcal{Y}$. That is,

$$\Pi_a(u, v) = \{\pi | \pi \text{ is a measure on } \mathcal{X} \times \mathcal{Y} \text{ such that } \pi(A \times \mathcal{Y}) = u(A) \\ \text{and } \pi(\mathcal{X} \times B) = v(B), \text{ for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}.\}$$

[1] Monge (1781)

Transferable Utility Matching and Optimal Transport

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty finite sets, u and v be probability measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi_a(u, v)$ the set of measures on $\mathcal{X} \times \mathcal{Y}$ that has the marginals u on $\mathcal{X}$ and v on $\mathcal{Y}$. That is,

$$\Pi_a(u, v) = \{\pi | \pi \text{ is a measure on } \mathcal{X} \times \mathcal{Y} \text{ such that } \pi(A \times \mathcal{Y}) = u(A) \\ \text{and } \pi(\mathcal{X} \times B) = v(B), \text{ for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}.\}$$

Given a continuous (or lower semi-continuous) function f, the optimal transport [1] minimization problem is to find a minimizer of the following problem:

$$\inf_{\pi \in \Pi_a(u,v)} \pi(f) := \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y)\pi(\{(x, y)\}). \tag{1}$$

[1] Monge (1781)

Transferable Utility Matching and Optimal Transport

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty finite sets, u and v be probability measures on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi_a(u, v)$ the set of measures on $\mathcal{X} \times \mathcal{Y}$ that has the marginals u on $\mathcal{X}$ and v on $\mathcal{Y}$. That is,

$$\Pi_a(u, v) = \{\pi | \pi \text{ is a measure on } \mathcal{X} \times \mathcal{Y} \text{ such that } \pi(A \times \mathcal{Y}) = u(A) \\ \text{and } \pi(\mathcal{X} \times B) = v(B), \text{ for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}.\}$$

Given a continuous (or upper semi-continuous) function g, the transferable utility matching (optimal transport maximization) problem is to find a maximizer of

$$\sup_{\pi \in \Pi_a(u,v)} \pi(g). \quad (2)$$

[1] Monge (1781)

Dual problem

- It is well known [1] that

$$\inf_{\pi \in \Pi_a(u,v)} \pi(f) = \sup_{\phi \oplus \psi \leq f} \sum_{x \in \mathcal{X}} \phi(x) u(\{x\}) + \sum_{y \in \mathcal{Y}} \psi(y) v(\{y\}), \qquad (3)$$

where $\phi : \mathcal{X} \to \mathbf{R}$ and $\psi : \mathcal{Y} \to \mathbf{R}$.

[1] Kantorovich (1942, 1948)

[2] Gale and Shapley (1962)

Dual problem

- It is well known [1] that

$$\inf_{\pi \in \Pi_a(u,v)} \pi(f) = \sup_{\phi \oplus \psi \leq f} \sum_{x \in \mathcal{X}} \phi(x) u(\{x\}) + \sum_{y \in \mathcal{Y}} \psi(y) v(\{y\}), \quad (3)$$

where $\phi : \mathcal{X} \to \mathbf{R}$ and $\psi : \mathcal{Y} \to \mathbf{R}$.

- Similarly,

$$\sup_{\pi \in \Pi_a(u,v)} \pi(g) = \inf_{\phi \oplus \psi \geq g} \sum_{x \in \mathcal{X}} \phi(x) u(\{x\}) + \sum_{y \in \mathcal{Y}} \psi(y) v(\{y\}) \quad (4)$$

[1] Kantorovich (1942, 1948)

[2] Gale and Shapley (1962)

Dual problem

- It is well known [1] that

$$\inf_{\pi \in \Pi_a(u,v)} \pi(f) = \sup_{\phi \oplus \psi \leq f} \sum_{x \in \mathcal{X}} \phi(x) u(\{x\}) + \sum_{y \in \mathcal{Y}} \psi(y) v(\{y\}), \qquad (3)$$

where $\phi : \mathcal{X} \to \mathbf{R}$ and $\psi : \mathcal{Y} \to \mathbf{R}$.

- Similarly,

$$\sup_{\pi \in \Pi_a(u,v)} \pi(g) = \inf_{\phi \oplus \psi \geq g} \sum_{x \in \mathcal{X}} \phi(x) u(\{x\}) + \sum_{y \in \mathcal{Y}} \psi(y) v(\{y\}) \qquad (4)$$

- This connects to the notion of *stable matching*[2].

[1] Kantorovich (1942, 1948)
[2] Gale and Shapley (1962)

Nonadditive Measures

Definition (Capacity)

Let $\mathcal{Z}$ be a nonempty finite set, and let $2^{\mathcal{Z}}$ be the collection of all of its subsets. A function $\gamma : 2^{\mathcal{Z}} \to \mathbb{R}$ is called a *capacity*[a] if $\gamma(\emptyset) = 0$, and $A \subseteq B$ implies $\gamma(A) \leq \gamma(B)$ for any $A, B \subset \mathcal{Z}$,

[a] Montrucchio (2004)

Nonadditive Measures

Definition (Capacity)

Let $\mathcal{Z}$ be a nonempty finite set, and let $2^{\mathcal{Z}}$ be the collection of all of its subsets. A function $\gamma : 2^{\mathcal{Z}} \to \mathbb{R}$ is called a *capacity* if $\gamma(\emptyset) = 0$, and $A \subseteq B$ implies $\gamma(A) \leq \gamma(B)$ for any $A, B \subset \mathcal{Z}$, *normalized* if $\gamma(\mathcal{Z}) = 1$.

Nonadditive Measures

Definition (Capacity)

Let $\mathcal{Z}$ be a nonempty finite set, and let $2^{\mathcal{Z}}$ be the collection of all of its subsets. A function $\gamma : 2^{\mathcal{Z}} \to \mathbb{R}$ is called a *capacity* if $\gamma(\emptyset) = 0$, and $A \subseteq B$ implies $\gamma(A) \leq \gamma(B)$ for any $A, B \subset \mathcal{Z}$, *normalized* if $\gamma(\mathcal{Z}) = 1$.

Definition (Choquet integral)

Let $\mathcal{Z}$ be a nonempty finite set, γ be a capacity on $\mathcal{Z}$, and $f : \mathcal{Z} \to \mathbb{R}_+$ be a nonnegative function on $\mathcal{Z}$. The Choquet integral of f with respect to γ is defined to be:

$$\gamma(f) := \int_0^\infty \gamma(\{f \geq t\})\, dt. \tag{5}$$

For a function $g : \mathcal{Z} \to \mathbb{R}$ that is not necessarily non-negative, then

$$\gamma(g) := \int_0^\infty \gamma(\{g \geq t\})\, dt + \int_{-\infty}^0 (\gamma(\{g \geq t\}) - \gamma(\mathcal{Z}))\, dt. \tag{6}$$

Optimal Transport with capacity marginals

let μ and ν be two normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi(\mu, \nu)$ the set of capacities on $\mathcal{X} \times \mathcal{Y}$ that has the marginals μ on $\mathcal{X}$ and ν on $\mathcal{Y}$. That is,

Optimal Transport with capacity marginals

let μ and ν be two normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi(\mu, \nu)$ the set of capacities on $\mathcal{X} \times \mathcal{Y}$ that has the marginals μ on $\mathcal{X}$ and ν on $\mathcal{Y}$. That is,

$$\Pi(\mu, \nu) = \{\pi | \pi \text{ is a capacity on } \mathcal{X} \times \mathcal{Y} \text{ such that } \pi(A \times \mathcal{Y}) = \mu(A) \\ \text{and } \pi(\mathcal{X} \times B) = \nu(B), \text{ for any } A \subseteq \mathcal{X}, B \subseteq \mathcal{Y}.\}$$

Optimal Transport with capacity marginals

let μ and ν be two normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$, respectively. Denote by $\Pi(\mu,\nu)$ the set of capacities on $\mathcal{X}\times\mathcal{Y}$ that has the marginals μ on $\mathcal{X}$ and ν on $\mathcal{Y}$. That is,

$$\Pi(\mu,\nu) = \{\pi|\pi \text{ is a capacity on } \mathcal{X}\times\mathcal{Y} \text{ such that } \pi(A\times\mathcal{Y}) = \mu(A) \\ \text{and } \pi(\mathcal{X}\times B) = \nu(B), \text{ for any } A\subseteq\mathcal{X}, B\subseteq\mathcal{Y}.\}$$

Given a continuous function f, the Optimal Transport problem on capacities aims to find optimizers of

$$\inf_{\pi\in\Pi(\mu,\nu)} \pi(f), \tag{7}$$

and

$$\sup_{\pi\in\Pi(\mu,\nu)} \pi(f). \tag{8}$$

Here $\pi(f)$ is the Choquet integral.

The ceiling and floor envelopes

Definition (floor and ceiling envelopes)

Let $\mathcal{Z}$ be a nonempty finite set, and let $\mathcal{G} \subseteq 2^{\mathcal{Z}}$ be a collection of subsets containing $\mathcal{Z}$ and $\emptyset$. Suppose that a function $G : \mathcal{G} \to \mathbb{R}_+$ satisfies $G(\emptyset) = 0$, and $G(A) \leq G(B)$ whenever $A, B \in \mathcal{G}$, $A \subseteq B$.

The ceiling and floor envelopes

Definition (floor and ceiling envelopes)

Let $\mathcal{Z}$ be a nonempty finite set, and let $\mathcal{G} \subseteq 2^{\mathcal{Z}}$ be a collection of subsets containing $\mathcal{Z}$ and $\emptyset$. Suppose that a function $G : \mathcal{G} \to \mathbb{R}_+$ satisfies $G(\emptyset) = 0$, and $G(A) \leq G(B)$ whenever $A, B \in \mathcal{G}$, $A \subseteq B$. The capacity on $\mathcal{Z}$ defined by:

$$G^*(B) = \inf_{B \subseteq A} G(A), \text{ for all } B \in 2^{\mathcal{Z}} \tag{9}$$

is called the ceiling envelope of G.

The ceiling and floor envelopes

Definition (floor and ceiling envelopes)

Let $\mathcal{Z}$ be a nonempty finite set, and let $\mathcal{G} \subseteq 2^{\mathcal{Z}}$ be a collection of subsets containing $\mathcal{Z}$ and $\emptyset$. Suppose that a function $G : \mathcal{G} \to \mathbb{R}_+$ satisfies $G(\emptyset) = 0$, and $G(A) \leq G(B)$ whenever $A, B \in \mathcal{G}$, $A \subseteq B$.
The capacity on $\mathcal{Z}$ defined by:

$$G^*(B) = \inf_{B \subseteq A} G(A), \text{ for all } B \in 2^{\mathcal{Z}} \tag{9}$$

is called the ceiling envelope of G.
The capacity defined by:

$$G_*(B) = \sup_{A \subseteq B} G(A), \text{ for all } B \in 2^{\mathcal{Z}} \tag{10}$$

is called the floor envelope of G.

$\Pi(\mu, \nu)$ is nonempty

Definition ($\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ and $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$)

Given nonempty finite sets $\mathcal{X}, \mathcal{Y}$, we define $\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ to be the collection of all subsets of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.

$\Pi(\mu, \nu)$ is nonempty

Definition ($\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ and $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$)

Given nonempty finite sets $\mathcal{X}, \mathcal{Y}$, we define $\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ to be the collection of all subsets of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.
We define $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$ to be the collection of all sets either of the form $\mathcal{X} \times B$ with $B \subseteq \mathcal{Y}$ or $A \times \mathcal{Y}$ with $A \subseteq \mathcal{X}$.

$\Pi(\mu, \nu)$ is nonempty

Definition ($\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ and $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$)

Given nonempty finite sets $\mathcal{X}, \mathcal{Y}$, we define $\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ to be the collection of all subsets of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.
We define $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$ to be the collection of all sets either of the form $\mathcal{X} \times B$ with $B \subseteq \mathcal{Y}$ or $A \times \mathcal{Y}$ with $A \subseteq \mathcal{X}$.

Given nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ and capacities μ on $\mathcal{X}$ and ν on $\mathcal{Y}$, we can define the function $G : \mathcal{P}_{\mathcal{X},\mathcal{Y}} \to \mathbb{R}_+$ by $G(A \times B) = \mu(A) \cdot \nu(B)$ for $A \times B \in \mathcal{P}_{\mathcal{X},\mathcal{Y}}$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.

$\Pi(\mu, \nu)$ is nonempty

Definition ($\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ and $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$)

Given nonempty finite sets $\mathcal{X}, \mathcal{Y}$, we define $\mathcal{P}_{\mathcal{X},\mathcal{Y}}$ to be the collection of all subsets of $\mathcal{X} \times \mathcal{Y}$ of the form $A \times B$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.
We define $\mathcal{P}^*_{\mathcal{X},\mathcal{Y}}$ to be the collection of all sets either of the form $\mathcal{X} \times B$ with $B \subseteq \mathcal{Y}$ or $A \times \mathcal{Y}$ with $A \subseteq \mathcal{X}$.

Given nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ and capacities μ on $\mathcal{X}$ and ν on $\mathcal{Y}$, we can define the function $G : \mathcal{P}_{\mathcal{X},\mathcal{Y}} \to \mathbb{R}_+$ by $G(A \times B) = \mu(A) \cdot \nu(B)$ for $A \times B \in \mathcal{P}_{\mathcal{X},\mathcal{Y}}$ with $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$.
Both the ceiling envelope G^* and the floor envelope G_* are capacites in $\Pi(\mu, \nu)$, showing in particular that

$\Pi(\mu, \nu)$ is always nonempty.

Characterization

Definition (π^* and π_*)

For each $A \subseteq \mathcal{X} \times \mathcal{Y}$, define the following:

$$\pi^*(A) = \sup_{\pi \in \Pi(\mu,\nu)} \pi(A), \quad \pi_*(A) = \inf_{\pi \in \Pi(\mu,\nu)} \pi(A). \tag{11}$$

Characterization

Definition (π^* and π_*)

For each $A \subseteq \mathcal{X} \times \mathcal{Y}$, define the following:

$$\pi^*(A) = \sup_{\pi \in \Pi(\mu,\nu)} \pi(A), \quad \pi_*(A) = \inf_{\pi \in \Pi(\mu,\nu)} \pi(A). \tag{11}$$

Theorem (Ghossoub-Saunders-Z., 2022)

$\min_{\pi \in \Pi(\mu,\nu)} \pi(f) = \pi_*(f)$ *and* $\max_{\pi \in \Pi(\mu,\nu)} \pi(f) = \pi^*(f)$.

Characterization

For a set $M \subseteq \mathcal{X} \times \mathcal{Y}$, define

$$M_{\mathcal{X}} := \{x \in \mathcal{X} : \exists z = (x, y) \in M\}, \tag{12}$$

Characterization

For a set $M \subseteq \mathcal{X} \times \mathcal{Y}$, define

$$M_{\mathcal{X}} := \{x \in \mathcal{X} : \exists z = (x, y) \in M\}, \tag{12}$$

and

$$\tilde{M}_{\mathcal{X}} := \{x \in \mathcal{X} : (x, y) \in M, \ \ \forall y \in \mathcal{Y}\} = ((M^c)_{\mathcal{X}})^c, \tag{13}$$

with a similar definition for $M_{\mathcal{Y}}$ and $\tilde{M}_{\mathcal{Y}}$.

Characterization

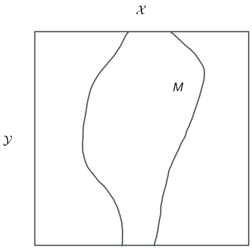
For a set $M \subseteq \mathcal{X} \times \mathcal{Y}$, define

$$M_{\mathcal{X}} := \{x \in \mathcal{X} : \exists z = (x, y) \in M\}, \tag{12}$$

and

$$\tilde{M}_{\mathcal{X}} := \{x \in \mathcal{X} : (x, y) \in M, \ \ \forall y \in \mathcal{Y}\} = ((M^c)_{\mathcal{X}})^c, \tag{13}$$

with a similar definition for $M_{\mathcal{Y}}$ and $\tilde{M}_{\mathcal{Y}}$.



Characterization

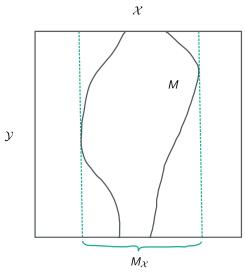
For a set $M \subseteq \mathcal{X} \times \mathcal{Y}$, define

$$M_{\mathcal{X}} := \{x \in \mathcal{X} : \exists z = (x, y) \in M\}, \tag{12}$$

and

$$\tilde{M}_{\mathcal{X}} := \{x \in \mathcal{X} : (x, y) \in M, \ \ \forall y \in \mathcal{Y}\} = ((M^c)_{\mathcal{X}})^c, \tag{13}$$

with a similar definition for $M_{\mathcal{Y}}$ and $\tilde{M}_{\mathcal{Y}}$.



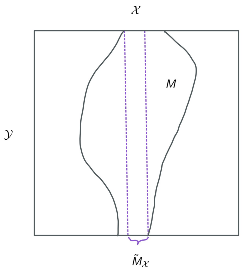
For a set $M \subseteq \mathcal{X} \times \mathcal{Y}$, define

$$M_{\mathcal{X}} := \{x \in \mathcal{X} : \exists z = (x, y) \in M\}, \tag{12}$$

and

$$\tilde{M}_{\mathcal{X}} := \{x \in \mathcal{X} : (x, y) \in M, \ \ \forall y \in \mathcal{Y}\} = ((M^c)_{\mathcal{X}})^c, \tag{13}$$

with a similar definition for $M_{\mathcal{Y}}$ and $\tilde{M}_{\mathcal{Y}}$.



Characterization

Recall

$\max_{\pi \in \Pi(\mu,\nu)} \pi(f) = \pi^*(f)$ and $\min_{\pi \in \Pi(\mu,\nu)} \pi(f) = \pi_*(f)$.

Characterization

Recall

$\max_{\pi\in\Pi(\mu,\nu)}\pi(f)=\pi^*(f)$ and $\min_{\pi\in\Pi(\mu,\nu)}\pi(f)=\pi_*(f)$.

Theorem (Ghossoub-Saunders-Z., 2022)

For any set $N\subseteq\mathcal{X}\times\mathcal{Y}$,

$$\pi^*(N)=\min(\mu(N_{\mathcal{X}}),\nu(N_{\mathcal{Y}})),$$

Characterization

Recall

$\max_{\pi\in\Pi(\mu,\nu)} \pi(f) = \pi^*(f)$ and $\min_{\pi\in\Pi(\mu,\nu)} \pi(f) = \pi_*(f)$.

Theorem (Ghossoub-Saunders-Z., 2022)

For any set $N \subseteq \mathcal{X} \times \mathcal{Y}$,

$$\pi^*(N) = \min(\mu(N_{\mathcal{X}}), \nu(N_{\mathcal{Y}})),$$
$$\pi_*(N) = \max(\mu(\tilde{N}_{\mathcal{X}}), \nu(\tilde{N}_{\mathcal{Y}})).$$

The cores

Definition (core)

Let γ be a normalized capacity on $\mathcal{Z}$. The core of γ is the set $\mathcal{C}(\gamma)$ of all probability measures v on $2^{\mathcal{Z}}$ such that $v(A) \geq \gamma(A)$ for all $A \in 2^{\mathcal{Z}}$.

The cores

Definition (core)

Let γ be a normalized capacity on $\mathcal{Z}$. The core of γ is the set $\mathcal{C}(\gamma)$ of all probability measures v on $2^{\mathcal{Z}}$ such that $v(A) \geq \gamma(A)$ for all $A \in 2^{\mathcal{Z}}$.

Proposition

Let μ and ν be capacities on nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then the following are equivalent.

- *Both μ and ν have nonempty cores (i.e. $\mathcal{C}(\mu) \neq \emptyset$ and $\mathcal{C}(\nu) \neq \emptyset$).*
- *There exists $\pi \in \Pi(\mu, \nu)$ with nonempty core.*

The cores

Definition (core)

Let γ be a normalized capacity on $\mathcal{Z}$. The core of γ is the set $\mathcal{C}(\gamma)$ of all probability measures v on $2^{\mathcal{Z}}$ such that $v(A) \geq \gamma(A)$ for all $A \in 2^{\mathcal{Z}}$.

Proposition

Let μ and ν be capacities on nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then the following are equivalent.

- *Both μ and ν have nonempty cores (i.e. $\mathcal{C}(\mu) \neq \emptyset$ and $\mathcal{C}(\nu) \neq \emptyset$).*
- *There exists $\pi \in \Pi(\mu, \nu)$ with nonempty core.*

Proposition

(a) *If $\mathcal{C}(\pi^*) \neq \emptyset$, then $\mathcal{C}(\pi) \neq \emptyset$ for all $\pi \in \Pi(\mu, \nu)$.*

The cores

Definition (core)

Let γ be a normalized capacity on $\mathcal{Z}$. The core of γ is the set $\mathcal{C}(\gamma)$ of all probability measures v on $2^{\mathcal{Z}}$ such that $v(A) \geq \gamma(A)$ for all $A \in 2^{\mathcal{Z}}$.

Proposition

Let μ and ν be capacities on nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then the following are equivalent.

- *Both μ and ν have nonempty cores (i.e. $\mathcal{C}(\mu) \neq \emptyset$ and $\mathcal{C}(\nu) \neq \emptyset$).*
- *There exists $\pi \in \Pi(\mu, \nu)$ with nonempty core.*

Proposition

(a) *If $\mathcal{C}(\pi^*) \neq \emptyset$, then $\mathcal{C}(\pi) \neq \emptyset$ for all $\pi \in \Pi(\mu, \nu)$.*

(b) *If $\mathcal{C}(\pi_*) = \emptyset$, then $\mathcal{C}(\pi) = \emptyset$ for all $\pi \in \Pi(\mu, \nu)$.*

The cores

Definition (core)

Let γ be a normalized capacity on $\mathcal{Z}$. The core of γ is the set $\mathcal{C}(\gamma)$ of all probability measures v on $2^{\mathcal{Z}}$ such that $v(A) \geq \gamma(A)$ for all $A \in 2^{\mathcal{Z}}$.

Proposition

Let μ and ν be capacities on nonempty finite sets $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then the following are equivalent.

- *Both μ and ν have nonempty cores (i.e. $\mathcal{C}(\mu) \neq \emptyset$ and $\mathcal{C}(\nu) \neq \emptyset$).*
- *There exists $\pi \in \Pi(\mu, \nu)$ with nonempty core.*

Proposition

(a) *If $\mathcal{C}(\pi^*) \neq \emptyset$, then $\mathcal{C}(\pi) \neq \emptyset$ for all $\pi \in \Pi(\mu, \nu)$.*

(b) *If $\mathcal{C}(\pi_*) = \emptyset$, then $\mathcal{C}(\pi) = \emptyset$ for all $\pi \in \Pi(\mu, \nu)$.*

(c) *In particular, $\mathcal{C}(\pi_*) \neq \emptyset$ iff $\mathcal{C}(\mu) \neq \emptyset$ and $\mathcal{C}(\nu) \neq \emptyset$.*

Cores of the optimal solutions

However, $\mathcal{C}(\pi^*)$ is typically empty.

Proposition (Core of π^*)

Suppose that μ and ν are normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$, and $|\mathcal{X}| \geq 2$, $|\mathcal{Y}| \geq 2$. Then $\mathcal{C}(\pi^*) = \emptyset$.

Cores of the optimal solutions

However, $\mathcal{C}(\pi^*)$ is typically empty.

Proposition (Core of π^*)

Suppose that μ and ν are normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$, and $|\mathcal{X}| \geq 2$, $|\mathcal{Y}| \geq 2$. Then $\mathcal{C}(\pi^*) = \emptyset$.

Proposition (Core of π_*)

If u is a probability measure on $\mathcal{X}$ and v is a probability measure on $\mathcal{Y}$, denote by $\Pi_a(u, v)$ the set of all probability measures on $\mathcal{X} \times \mathcal{Y}$ with marginal distributions u and v respectively. Let μ and ν be normalized capacities on $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then:

$$\mathcal{C}(\pi_*) = \cup_{u \in \mathcal{C}(\mu), v \in \mathcal{C}(\nu)} \Pi_a(u, v). \tag{14}$$

The Möbius transform

Definition

The Möbius transform of a capacity γ is defined as:

$$m^{\gamma}(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \gamma(B). \tag{15}$$

The Choquet integral of f with respect to γ can be represented as:

$$\gamma(f) = \sum_{B \subseteq \mathcal{X}} K_f(B)\gamma(B) \tag{16}$$

$$= \sum_{A \subseteq \mathcal{X}} m^{\gamma}(A) \min_{x \in A} f(x) \tag{17}$$

with:

$$K_f(B) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \min_{x \in A} f(x).$$

Duality I

Theorem (Duality I, Ghossoub-Saunders-Z., 2022)

The dual of the minimization Optimal Transport problem is equivalent to

$$\max_{\hat{\varphi},\hat{\psi},\hat{\rho}} \sum_{G\subseteq\mathcal{X}} \hat{\varphi}(G)\mu(G) + \sum_{F\subseteq\mathcal{Y}} \hat{\psi}(F)\nu(F) \tag{18}$$

subject to

$$\begin{aligned}
\hat{\varphi}(G) - \sum_{w\notin G\times\mathcal{Y}} \hat{\rho}(G\times\mathcal{Y}, w) + \sum_{w\in G\times\mathcal{Y}} \hat{\rho}((G\times\mathcal{Y})\setminus\{w\}, w) &= K_c(G\times\mathcal{Y}), \quad \emptyset \neq G \subsetneqq \mathcal{X};\\
\hat{\psi}(F) - \sum_{w\notin \mathcal{X}\times F} \hat{\rho}(\mathcal{X}\times F, w) + \sum_{w\in \mathcal{X}\times F} \hat{\rho}((\mathcal{X}\times F)\setminus\{w\}, w) &= K_c(\mathcal{X}\times F), \quad \emptyset \neq F \subsetneqq \mathcal{Y};\\
\hat{\varphi}(\mathcal{X}) + \hat{\psi}(\mathcal{Y}) + \sum_{w} \hat{\rho}((\mathcal{X}\times\mathcal{Y})\setminus\{w\}, w) &= K_c(\mathcal{X}\times\mathcal{Y});\\
-\sum_{w\notin B} \hat{\rho}(B, w) + \sum_{w\in B} \hat{\rho}(B\setminus\{w\}, w) &= K_c(B), \quad B \notin \mathcal{P}^*_{\mathcal{X},\mathcal{Y}};\\
\hat{\rho} &\geq 0.
\end{aligned} \tag{19}$$

Duality II

Theorem (Duality II, Ghossoub-Saunders-Z., 2022)

The dual of the minimization Optimal Transport problem is also equivalent to

$$\max_{L_\varphi, L_\psi, \hat{\rho}} \sum_{G \subseteq \mathcal{X}} L_\varphi(G) m^\mu(G) + \sum_{F \subseteq \mathcal{Y}} L_\psi(F) m^\nu(F) \tag{20}$$

$$L_\varphi(A_\mathcal{X}) + L_\psi(A_\mathcal{Y}) + \sum_{D \supseteq A} \sum_{w \in A} \hat{\rho}(D \setminus \{w\}, w) = \min_{(x,y) \in A} c(x,y), \quad \emptyset \neq A \subseteq \mathcal{X} \times \mathcal{Y}; \tag{21}$$
$$\hat{\rho} \geq 0.$$

Conclusion and Future work

- Studied the *Transferable Utility Matching* and *Optimal Transport* problems with capacity marginals
- Provided characterizations of the optimal solutions and their cores
- Built the duality theory
- Results on infinite spaces is still open

Thank you!

References

D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

Leonid V Kantorovich. On the translocation of masses. *C.R. (Doklady) Acad. Sci. URSS (N.S.)*, 37:199–201, 1942.

Leonid Vital'evich Kantorovich. On a problem of monge(in russian). *Uspekhi Math. Nauk.*, 3:225–226, 1948.

Gaspard Monge. Mémoire sur la théorie des déblais et des remblais. *Mem. Math. Phys. Acad. Royale Sci.*, pages 666–704, 1781.

M Marinacci-L Montrucchio. Introduction to the mathematics of ambiguity. *Uncertainty in Economic Theory, I. Gilboa (ed.)*, pages 46–107, 2004.